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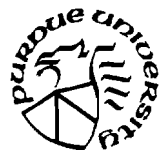
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FAST SOLUTION TECHNIQUES FOR A
CLASS OF OPTIMAL TRAJECTORY
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APPLICATIONS TO AUTOMATED
SPRAY COATING

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ABSTRACT

Optimal trajectory planning problems are often formulated as constrained variational problems. In general, solutions to variational problems are determined by appropriately discretizing the underlying objective functional and solving the resulting nonlinear differential equation(s) and/or nonlinear programming problem(s) numerically. These general solution techniques often require a significant amount of time to be computed, and therefore are of limited value when optimal trajectories need to be frequently computed and/or re-computed. In this paper, a realistic class of optimal trajectory planning problems is defined for which the existence of fast numerical solution techniques are demonstrated. To illustrate the practicality of this class of trajectory planning problems and the proposed solution techniques, three optimal trajectory planning problems for spray coating applications are formulated and solved. Based on the proposed discretization technique, it is shown that these problems can be reduced to either a linear program or a quadratic program, which are readily solved. In contrast, using the standard discretization of these problems generally leads to nonconvex nonlinear programming problems that require a significant amount of computation to arrive at a (possibly) locally optimal solution.

I. INTRODUCTION

This paper addresses solution techniques for a class of trajectory planning problems that arise in manufacturing applications. The discussion is motivated by a particular problem in spray coating applications, where the objective is to determine the optimal time profile for a spray coating applicator that is constrained to traverse a specified spatial path.

In large-scale production lines, spray coating applicators are attached to robotic manipulators that move the applicator around the surface to be coated. Experienced operators of such systems can often provide good choices for the spatial path of the robot's end-effector. An operator typically "teaches" the robot a desired spatial path by moving the end-effector around the part to be coated while the robot's control computer records position and orientation information [12]. A less intuitive issue (than selecting effective spatial paths) is to decide how to traverse a given spatial path temporally (i.e., with respect to time). In general, the accumulated film thickness of a target area is proportional to the amount of time spent spraying the area. Therefore, moving the applicator more slowly over certain regions may be called for if the spatial path is such that there is very little accumulation contributed to the area by other positions on the path. There can be trade-offs between achieving uniform coatings and minimizing wasted paint, especially when traversing near the edges of a part. Studies into these types of problems have been conducted in the past [1, 3, 11].

The studies in [1, 3] discuss general methods that are applicable for automatically determining both the spatial and temporal components of the applicator's trajectory using nonlinear programming methods. In the present paper, the focus is on determining the optimal time profile of an applicator that is constrained to traverse a specified spatial path. Although the "time and space" formulations of the past (i.e., [1, 3]) can

be applied to the restricted problem of finding the optimal time profile for traversing a specified spatial path, they still generally result in nonlinear (and nonconvex) programming problems. In contrast, an alternate formulation is proposed here for the restricted problem that results in either linear or quadratic programming problems, depending on the specific objective function assumed.

It is assumed that the positions along a spatial path are characterized by a continuous vector function $\mathbf{p}(\lambda)$, where the elements of $\mathbf{p}(\lambda)$ define the coordinates of the applicator as a function of the scalar parameter λ . It is further assumed that the spatial path is parameterized by arc length, which means that a unit change in the parameterizing variable λ results in a unit change in curve length along the path [2]. For this type of parameterization, $\lambda \in [0, L]$, where L is the total length of the path. To model the motion of the applicator along a parameterized path during a time interval $[0, T]$, the scalar quantity λ is replaced by a scalar function of time $\psi(t)$, where $\psi : [0, T] \rightarrow [0, L]$. Therefore, the position of the applicator at a given instant of time t is specified by $\mathbf{p}(\psi(t))$. The function $\psi(t)$ is referred to as the time profile of the applicator.

In general path planning problems, the objective is to determine $\mathbf{p}(\psi(t))$, i.e., both $\mathbf{p}(\cdot)$ and $\psi(\cdot)$, to optimize a given performance index. Such problems are typically formulated as constrained variational problems, where the objective is to minimize the cost functional that depends on $\mathbf{p}(\psi(t))$. In this paper, the spatial path is assumed to be given, therefore, the only unknown within the cost functional is the scalar function $\psi(t)$.

The cost functional and any constraint functionals for spray coating are typically associated with one or more process performance metrics such as painting time, variation in film thickness, average film thickness, expended paint, and transfer efficiency. When the spatial path is specified, the problem is to determine the function $\psi(t)$ to

satisfy the performance constraints and optimize a specified performance index associated with the cost functional. The following optimization problems are considered in this paper: (1) minimize painting time subject to achieving a specified average thickness; (2) minimize variation in film thickness subject to achieving a specified average thickness, and (3) minimize variation in film thickness subject to achieving a specified average thickness and an upper bound on painting time. Although the paper addresses methods for these specific problems, the framework developed can also be applied to other performance objectives.

The remainder of the paper is organized in the following manner. Section II outlines some basic assumptions and definitions, and expressions for film thickness (for each surface point), average film thickness, and the variation in film thickness are derived. In Section III, two different approximate expressions are developed for the film thickness function. The first expression, called the standard approximation, has been used in the past (e.g., [1, 11]). The second expression, called the alternate approximation, is the key to formulating the proposed methods for solving the three optimization problems under consideration. Each of the three optimization problems are formulated using both approximations in Section IV. It is shown that the standard approximation generally leads to nonlinear programming problems, while the alternate approximation yields linear or quadratic programming problems. Section V includes numerical studies to illustrate the computational advantages of the proposed alternate formulation over the standard formulations for the three optimization problems.

II. BASIC ASSUMPTIONS AND DEFINITIONS

The surface to be coated is defined by a set of points $\mathbf{S} \subset \mathbf{R}^3$. The set of points along the parameterized spatial path $\mathbf{p}(\lambda)$ (which defines the positions at which the

applicator is constrained to be located) is defined by $\mathcal{A}_p = \{a : a = p(\lambda), \lambda \in [0, L]\}$. It is assumed that the orientation of the applicator is specified for each point in this set. A typical specification in spray coating is to orient the applicator normal to the surface that is to be coated. A mapping, $\mathbf{f} : \mathbf{S} \times \mathcal{A}_p \rightarrow \mathcal{R}^+$ is assumed, which defines the rate of film accumulation at each surface point $\mathbf{s} \in \mathbf{S}$ for each possible location of the applicator $a \in \mathcal{A}_p$. Therefore, $\mathbf{f}(\mathbf{s}, p(\psi(t)))$ represents the rate of film accumulation for each surface point $\mathbf{s} \in \mathbf{S}$ at time t , where the applicator traverses the parameterized spatial path according to $p(\psi(t))$, and $\psi : [0, T] \rightarrow [0, L]$.

The film thickness (for each surface point \mathbf{s}) accumulated over the time period $[0, T]$ is denoted by $F(\mathbf{s}, p(\cdot), \psi(\cdot), T)$ and is obtained by integrating the assumed film accumulation rate function over the time period $[0, T]$:

$$F(\mathbf{s}, p(\cdot), \psi(\cdot), T) = \int_0^T f(\mathbf{s}, p(\psi(t))) dt. \quad (1)$$

Thus, there are three parameters that affect the accumulated film thickness at each point: the parameterized spatial path, $p(\cdot)$; the time profile for traversing the spatial path, $\psi(\cdot)$; and painting time, T . More general models could include the effect of other parameters such as shaping air pressure and paint flow rate [9]. This paper, however, does not discuss the control of these types of parameters because they are generally difficult and/or impractical to accurately control (i.e., vary) over time.

The basic assumption made here is that, for a given a set of distinct positions of the applicator along a specified path, the corresponding film accumulation rate at the surface points (characterized by the mapping \mathbf{f}) is known. This mapping can be based on theoretical models and/or be derived from empirical data collected through off-line experimentation. For example, film thickness measurements could be taken after spraying paint for a small (and known) amount of time from each point along the spatial path. (Both wet- and dry-film gauges can be used to measure film thickness;

for a detailed description of such devices, refer to [12].)

Two important measures of quality that are used in the optimization problems considered in this paper are: (1) the average film thickness and (2) the variation in film thickness over the surface. These quantities, which characterize the deposition of paint over a surface, depend on the film thickness function given in Equation 1.

The average film thickness accumulated over a surface is defined by the total volume of paint deposited on the surface divided by the area of the surface. Therefore, the formula for average film thickness, denoted by $G(\mathbf{p}(\cdot), \psi(\cdot), T)$, is obtained by integrating the expression for film thickness over the entire surface and dividing by the area of the surface:

$$G(\mathbf{p}(\cdot), \psi(\cdot), T) = \frac{1}{A_S} \int_S F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) d\mathbf{s}, \quad (2)$$

where,

$$A_S = \int_S d\mathbf{s}. \quad (3)$$

The variation in film thickness, defined as the total mean squared error between the actual thickness and the average thickness, is a measure of uniformity of the coating. Therefore, the formula for the variation in film thickness, denoted by $V(\mathbf{p}(\cdot), \psi(\cdot), T)$, is obtained by integrating the squared difference between the actual and the average thickness over the entire surface and dividing the area of the surface:

$$V(\mathbf{p}(\cdot), \psi(\cdot), T) = \frac{1}{A_S} \int_S (F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) - G(\mathbf{p}(\cdot), \psi(\cdot), T))^2 d\mathbf{s}. \quad (4)$$

The expression for film thickness (Equation 1) appears in both of these performance indicators (Equations 2 and 4). In optimization problems where the objective and/or constraints are based on expressions such as these, which depend on the film thickness function, determining an appropriate representation for the film thickness function in terms of $\psi(\cdot)$ is important. This issue is studied in the next section.

III. APPROXIMATE EXPRESSIONS FOR THE FILM THICKNESS FUNCTION

One difficulty in solving optimization problems involving the film thickness function is due to the fact that, in many cases, analytical expressions for the film thickness function (in terms of $\psi(t)$) are either not possible to compute or difficult to determine. Computing the film thickness function involves the integration of the film accumulation rate function $f(\mathbf{s}, \mathbf{p}(\psi(t)))$, and this film accumulation rate function is typically a nonlinear function of $\psi(\cdot)$. An example of such a function is the bivariate Cauchy function, considered in [1].

By approximating the film thickness function using an appropriate discretization technique, the given variational problem in $\psi(\cdot)$ reduces to a finite dimensional optimization problem. A standard discrete approximation for the film thickness function is outlined in the next subsection. An alternate approach is then derived in Subsection III.B. The standard approach results in an expression that is nonlinear with respect to the associated discrete variables, while the alternate expression is linear with respect to its discrete set of variables. The linearity of the alternate expression for film thickness enables the corresponding expressions for average thickness and variation in film thickness to be expressed as linear and quadratic functions, respectively.

A. A Standard Approximation

The time profile function $\psi(\cdot)$ can be approximated with a piecewise constant function. The time interval $[0, T]$ is divided into N sub-intervals, where each sub-interval is of width $\Delta = T/N$. A box-car function $b_i(t)$ is defined over the i -th sub-interval, $i = 1, 2, \dots, N$, as follows:

$$b_i(t) = \begin{cases} 1 & \text{if } t \in [(i-1)\Delta, i\Delta] \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

The value of the function $\psi(t)$ at the center of each sub-interval is evaluated. These values are denoted by $\psi_1, \psi_2, \dots, \psi_N$. The function $\psi(t)$ is approximated by a function $\tilde{\psi}(t)$, defined as

$$\tilde{\psi}(t) = \sum_{i=1}^N \psi_i b_i(t) \approx \psi(t), \quad (6)$$

where $\tilde{\psi}(t)$ represents a piecewise constant approximation to the function $\psi(t)$. Let the N sampled values of the function be grouped into a vector $\Psi = [\psi_1, \psi_2, \dots, \psi_N]'$, where the prime denotes transpose. Substituting the expression for $\tilde{\psi}(t)$ into Equation 1 (in place of $\psi(t)$), the expression for the film thickness function is approximated by the following equation:

$$\tilde{F}(\mathbf{s}, \mathbf{p}(\cdot), \Psi, T) = \Delta \sum_{i=1}^N f(\mathbf{s}, \mathbf{p}(\psi_i)) \approx F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T). \quad (7)$$

Using this approximate expression for the film thickness function, the average thickness function $G(\mathbf{p}(\cdot), \psi(\cdot), T)$ can be approximated as

$$\tilde{G}(\mathbf{p}(\cdot), \Psi, T) = \frac{\Delta}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s} \approx G(\mathbf{p}(\cdot), \psi(\cdot), T). \quad (8)$$

With the approximations for the film thickness and average film thickness functions, the variation in film thickness can be approximated similarly. First, note that an equivalent general expression for the variation in film thickness of Equation 4 is

$$\mathcal{V}(\mathbf{p}(\cdot), \psi(\cdot), T) = \frac{1}{A_S} \int_S (F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T))^2 d\mathbf{s} - (G(\mathbf{p}(\cdot), \psi(\cdot), T))^2. \quad (9)$$

Thus, the approximation for the variation in film thickness is written as

$$\begin{aligned} \mathcal{V}(\mathbf{p}(\cdot), \Psi, T) &= \left(\Delta \sum_{i=1}^N f(\mathbf{s}, \mathbf{p}(\psi_i)) \right)^2 d\mathbf{s} - \left(\frac{\Delta}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s} \right)^2 \\ &\approx \mathcal{V}(\mathbf{p}(\cdot), \psi(\cdot), T). \end{aligned} \quad (10)$$

Equations 7, 8, and 10 are nonlinear expressions in the vector of variables \mathbf{p} , which represent approximations to the film thickness function, the average thickness

function, and the variation in film thickness, respectively. Therefore, a variational problem in $\psi(\cdot)$ involving any of these quantities can be reduced to a finite dimensional optimization problem in Ψ . Such finite dimensional optimization problems can be solved by nonlinear programming methods.

B. An Alternate Approximation

To reduce the complexity of computation generally associated with solving the nonlinear programming problems generated by the formulation of the previous subsection, an alternate formulation is developed for approximating the functions for film thickness, average thickness, and variation in film thickness. The proposed formulation is based on utilizing an alternate discretization of the time profile function. In this alternate approach, a finite number of evenly spaced points along the spatial path are considered and the amount of time spent at each of these spatial points are used as variables. This is in contrast to the discretization used in the previous subsection in which a finite number of evenly-spaced time instances are considered and the spatial positions for each of these time instants are used as variables. As shown in Figure 1, it is clear that if the time profile function $\psi(\cdot)$ is monotone, then the two discretization methods approach equivalence as the number of discrete sample points used by each approach is increased.

The alternate approach (Figure 1(b)) requires that $\psi(\cdot)$ be monotone in order to be well-defined, while the standard approach of the previous subsection is applicable for arbitrary $\psi(\cdot)$. From this observation, it would appear that the proposed alternate approach is not as general as the standard approach, because it can represent only monotone choices for $\psi(\cdot)$. However, a theorem is presented below that proves that for every time profile $\psi(\cdot)$, there exists a corresponding monotone time profile $\bar{\psi}(\cdot)$ for which the resulting film thickness functions generated by these two time profiles are

identical. Thus, the theorem proves that it is sufficient to consider only monotone time profile functions, which implies that the proposed alternate discretization scheme can be employed without loss of generality. The advantage of the proposed approach is that the approximate expressions for the average thickness function and the variation in film thickness reduce to linear and quadratic expressions, respectively.

Theorem 1: Given a spatial path parameterization $\mathbf{p}(\cdot)$, for every continuously differentiable time profile function $\psi(\cdot)$, $\psi : [0, T] \rightarrow [0, L]$, there exists a monotone time profile function $\bar{\psi}(t)$ such that

$$F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) = F(\mathbf{s}, \mathbf{p}(\cdot), \bar{\psi}(\cdot), T), \quad \text{for all } \mathbf{s} \in \mathcal{S}. \quad (11)$$

Proof: A key property of a monotone time profile function $\bar{\psi}(t)$ is that it is invertible (i.e., $\bar{\psi}^{-1}(\cdot)$ exists). Using a change of variables $\lambda = \bar{\psi}(t)$ enables the integration over time of Equation 1 to be replaced with an equivalent integral over space:

$$F(\mathbf{s}, \mathbf{p}(\cdot), \bar{\psi}(\cdot), T) = \int_0^L f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda} \right) d\lambda. \quad (12)$$

To prove the assertion of the theorem, the case of a non-monotone time profile function having two extreme points is analyzed. The case of having more than two extreme points follows by induction.

Consider the non-monotone time profile function shown in Figure 2. Although the function is not monotone over the entire interval $[0, T]$, the function is monotone in each of the segments $[0, t_1]$, $[t_1, t_2]$, and $[t_2, T]$. Let $\psi_1(t) = \psi_{[0, t_1]}(t)$, $\psi_2(t) = \psi_{[t_1, t_2]}(t)$, and $\psi_3(t) = \psi_{[t_2, T]}(t)$ denote the function $\psi(t)$ over the respective intervals. Each of these functions is monotone and hence invertible. Based on these three intervals, expressions for the film thickness function of Equation 1 can be written as

$$F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) = \int_0^{t_1} f(\mathbf{s}, \mathbf{p}(\psi_1(t))) dt + \int_{t_1}^{t_2} f(\mathbf{s}, \mathbf{p}(\psi_2(t))) dt + \int_{t_2}^T f(\mathbf{s}, \mathbf{p}(\psi_3(t))) dt. \quad (13)$$

Because each function $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ is monotone, the three integrals over time can be replaced with three spatial integrals by applying Equation 12:

$$\begin{aligned} F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) = & \int_0^{\lambda_1} f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_1^{-1}(\lambda)}{d\lambda} \right) d\lambda + \\ & \int_{\lambda_1}^{\lambda_2} f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_2^{-1}(\lambda)}{d\lambda} \right) d\lambda + \\ & \int_{\lambda_2}^L f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_3^{-1}(\lambda)}{d\lambda} \right) d\lambda. \end{aligned} \quad (14)$$

Rearranging the limits of integration into three non-overlapping intervals in the variable λ , the above equation can be written as

$$\begin{aligned} F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) = & \int_0^{\lambda_2} f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_1^{-1}(\lambda)}{d\lambda} \right) d\lambda + \\ & \int_{\lambda_2}^{\lambda_1} f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_1^{-1}(\lambda)}{d\lambda} - \frac{d\psi_2^{-1}(\lambda)}{d\lambda} + \frac{d\psi_3^{-1}(\lambda)}{d\lambda} \right) d\lambda + \\ & \int_{\lambda_1}^L f(\mathbf{s}, \mathbf{p}(\lambda)) \left(\frac{d\psi_3^{-1}(\lambda)}{d\lambda} \right) d\lambda. \end{aligned} \quad (15)$$

Define the function $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ in the following manner:

$$\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda} = \begin{cases} \left(\frac{d\psi_1^{-1}(\lambda)}{d\lambda} \right) & \text{for } \lambda \in (0, \lambda_2) \\ \left(\frac{d\psi_1^{-1}(\lambda)}{d\lambda} - \frac{d\psi_2^{-1}(\lambda)}{d\lambda} + \frac{d\psi_3^{-1}(\lambda)}{d\lambda} \right) & \text{for } \lambda \in (\lambda_2, \lambda_1) \\ \left(\frac{d\psi_3^{-1}(\lambda)}{d\lambda} \right) & \text{for } \lambda \in (\lambda_1, L). \end{cases} \quad (16)$$

Because $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ is non-negative for all $\lambda \in [0, L]$, $\bar{\psi}^{-1}(\lambda)$ is monotone, which implies $\bar{\psi}(t)$ is monotone in the interval $[0, T]$. ■

Theorem 1 provides the justification for the proposed alternate formulation. Because every non-monotone time profile has a corresponding monotone time profile that generates the same film thickness function, the search space for any associated optimization problem can be reduced to the set of monotone time profile functions. As a result, Equation 12 can be used to represent all possible film thickness functions using monotone time profiles, as opposed to Equation 1, which represents film thickness functions for arbitrary time profiles.

Instead of directly searching for the function $\psi(t)$ as required by Equation 1, the formulation of Equation 12 is based on determining a monotone function $\bar{\psi}^{-1}(\lambda)$. This is done by searching for an appropriate function $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ that is positive (which ensures that $\bar{\psi}^{-1}(\lambda)$ and $\bar{\psi}(t)$ are monotone). Denoting $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ by $\nu(\lambda)$, Equation 12 can be rewritten as

$$F(\mathbf{s}, \mathbf{p}(\cdot), \bar{\psi}(\cdot), T) = \int_0^L f(\mathbf{s}, \mathbf{p}(\lambda)) \nu(\lambda) d\lambda. \quad (17)$$

To approximate the integral of Equation 17, the spatial interval $[0, L]$ is divided into N sub-intervals, where each sub-interval is of width $\delta = L/N$. The box-car function $b_i(\lambda)$ (as defined in Equation 5) is used to define an approximate representation for $\nu(\lambda)$ as

$$\tilde{\nu}(\lambda) = \sum_{i=1}^N \nu_i b_i(\lambda) \approx \nu(\lambda), \quad (18)$$

where ν_i denotes the value of the function $\nu(\cdot)$ in the center of the i -th spatial sub-interval along the path. As the spatial path is also a function of λ , a similar approximation for the function $\mathbf{p}(\lambda)$ is defined by

$$\tilde{\mathbf{p}}(\lambda) = \sum_{i=1}^N \mathbf{p}_i b_i(\lambda) \approx \mathbf{p}(\lambda), \quad (19)$$

where the terms $\mathbf{p}_i = \mathbf{p}((i-1)\delta + \delta/2)$ are known because the spatial path $\mathbf{p}(\cdot)$ is assumed to be given. Thus, the alternate expression for the film thickness function in Equation 17 can be approximated using this discretized representation as

$$F(\mathbf{s}, \mathbf{p}(\cdot), \bar{\psi}(\cdot), T) \approx \sum_{i=1}^N f(\mathbf{s}, \mathbf{p}_i) \delta \nu_i. \quad (20)$$

The quantity $\delta \nu_i$ represents the time spent by the applicator in the i -th spatial sub-interval. This is because the values of ν_i represent the reciprocal of the applicator's speed over the i -th spatial sub-interval ($\nu(\lambda) = \frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda} = \frac{1}{\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}}$) and δ is the width of each spatial sub-interval. Denoting the quantity $\delta \nu_i$ by τ_i , the alternate expression for

the film thickness function is given by

$$\tilde{F}_a(\mathbf{s}, \mathbf{p}(\cdot), \Gamma, T) = \sum_{i=1}^N f(\mathbf{s}, \mathbf{p}_i) \tau_i \approx F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T), \quad (21)$$

where $\Gamma = [\tau_1, \tau_2, \dots, \tau_N]'$ represents the vector of discrete variables for the alternate approximation. The subscript "a" is used to distinguish this alternate expression from the standard approximation of Equation 7.

A comparison of Equations 7 and 21 illustrates the simplification that results from the alternate formulation. In Equation 7 (i.e., the standard approximation for the film thickness function), the quantity $f(\mathbf{s}, \mathbf{p}(\psi_i))$ is unknown, because it depends on the discrete variables ψ_i , which are to be determined. In contrast, in the alternate approximation of the film thickness function (Equation 21), the corresponding quantity, $f(\mathbf{s}, \mathbf{p}_i)$, is known because each spatial point \mathbf{p}_i , is known from the given parameterization of the path. The unknown variables in Equation 21 are the τ_i 's. Therefore, the alternate expression for the film thickness function is a linear function of these variables. Using the alternate expression for the film thickness function in Equation 21, the associated average thickness function is approximated as

$$\tilde{G}_a(\mathbf{p}(\cdot), \Gamma, T) = \sum_{i=1}^N \tau_i \frac{1}{A_S} \int_S f(\mathbf{s}, \mathbf{p}_i) d\mathbf{s} \approx G(\mathbf{p}(\cdot), \psi(\cdot), T). \quad (22)$$

By defining $\mathbf{g}_i = \frac{1}{A_S} \int_S f(\mathbf{s}, \mathbf{p}_i) d\mathbf{s}$, and denoting the vector of all \mathbf{g}_i 's by $\mathbf{g} = [g_1, g_2, \dots, g_N]'$, the approximate average thickness function can be expressed as

$$\tilde{G}_a(\mathbf{p}(\cdot), \Gamma, T) = \sum_{i=1}^N \mathbf{g}_i \tau_i = \mathbf{g}' \Gamma, \quad (23)$$

where the \mathbf{g}_i 's are constant coefficients.

Similarly, the variation in film thickness, $V(\mathbf{p}(\cdot), \psi(\cdot), T)$, as expressed in Equation 9, can be approximated as

$$\begin{aligned} \tilde{V}_a(\mathbf{p}(\cdot), \Gamma, T) &= \frac{1}{A_S} \int_S \left(\sum_{i=1}^N f(\mathbf{s}, \mathbf{p}_i) \tau_i \right)^2 d\mathbf{s} - \left(\sum_{i=1}^N \mathbf{g}_i \tau_i \right)^2 \\ &\approx V(\mathbf{p}(\cdot), \psi(\cdot), T). \end{aligned} \quad (24)$$

Note that the alternate expression for the variation in film thickness is a quadratic expression in the τ_i 's, in contrast to the generally nonlinear representation of Equation 10.

IV. THREE OPTIMIZATION PROBLEMS

The three optimization problems discussed in Section 1 are now formulated based on the standard and alternate approximations for film thickness, average thickness, and variation in film thickness developed in the previous section. The optimization problems considered are: (1) minimize painting time subject to achieving a specified average thickness; (2) minimize variation in film thickness subject to achieving a specified average thickness, and (3) minimize variation in film thickness subject to achieving a specified average thickness and an upper bound on painting time. Because achieving a specified average thickness is a common constraint, the three problems are referred to as minimum painting time, minimum variation, and time constrained minimum variation problems, respectively.

A. The Minimum Painting Time Problem

A.1 Standard Formulation

Given a parameterized spatial path $\mathbf{p}(\cdot)$, and an associated film accumulation rate function characterized by the mapping \mathbf{f} , the objective of the minimum painting time problem is to minimize the time T required to achieve a specified average thickness H over the given surface. Using the standard approximation, $T = NA$, therefore the problem is formulated as

$$\min_{[\psi_1, \psi_2, \dots, \psi_N]} \{ NA \} \quad (25)$$

$$\text{subject to } \tilde{G}(\mathbf{p}(\cdot), \mathbf{9}, N\Delta) = H \quad (26)$$

$$\text{and } 0 \leq \psi_i \leq L, \quad \forall i. \quad (27)$$

Applying Equation 8, the equality constraint is eliminated by expressing \mathbf{A} in terms of the variables $[\psi_1, \psi_2, \dots, \psi_N]$ as follows:

$$\Delta = \frac{H}{\frac{1}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s}}. \quad (28)$$

Thus, the problem of determining the minimum painting time reduces to determining ψ_i 's such that \mathbf{A} is minimized, which is represented as the following optimization:

$$\begin{aligned} \min_{[\psi_1, \psi_2, \dots, \psi_N]} & \left\{ \frac{NH}{\frac{1}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s}} \right\}, \\ \text{subject to } & 0 \leq \psi_i \leq L, \quad \forall i. \end{aligned} \quad (29)$$

This problem is a constrained nonlinear programming problem.

A.2 Alternate Formulation

In contrast to the above, suppose that the alternate approximation to the average thickness function is used (Equation 23). As the sum of the associated unknown variables (the τ_i 's) represents the total painting time T , the problem in this framework is formulated as

$$\min_{[\tau_1, \tau_2, \dots, \tau_N]} \left\{ \sum_{i=1}^N \tau_i \right\} \quad (31)$$

$$\text{subject to } \tilde{G}_a(\mathbf{p}(\cdot), \Gamma, T) = H \quad (32)$$

$$\text{and } \tau_i \geq 0, \quad \forall i. \quad (33)$$

Expressing the average thickness function in terms of Equation 23, the minimum painting time problem becomes a standard linear programming problem:

$$\min_{\Gamma} \{\mathbf{1}'\Gamma\} \quad (34)$$

$$\text{subject to } \mathbf{g}'\Gamma = H \quad (35)$$

$$\text{and } \Gamma \geq 0, \quad (36)$$

where $\mathbf{1} = [1, 1, \dots, 1]'$.

Unlike general linear programs that are usually solved using simplex methods, a simple closed-form solution can be formulated to the above problem. Suppose the maximum of the elements of the vector \mathbf{g} is at the q -th index, then the average thickness constraint can be rewritten as

$$\tau_q = \frac{1}{g_q} \left(H - \sum_{i \neq q}^N \tau_i g_i \right). \quad (37)$$

The value for τ_q in the cost function (Equations 31 and 34) can be substituted by Equation 37. Thus the painting time is represented as

$$\sum_{i=1}^N \tau_i = \sum_{i \neq q}^N \tau_i + \frac{1}{g_q} \left(H - \sum_{i \neq q}^N \tau_i g_i \right), \quad (38)$$

$$\Rightarrow \sum_{i=1}^N \tau_i = \sum_{i \neq q}^N \left(1 - \frac{g_i}{g_q} \right) \tau_i + \frac{H}{g_q}. \quad (39)$$

Observe that the equation for the total time is a sum of non-negative quantities, and thus the minimum cannot be less than zero. Thus, the solution is written as

$$\tau_i = \begin{cases} 0 & \text{if } i \neq q \\ \frac{H}{g_q} & \text{if } i = q. \end{cases} \quad (40)$$

The physical implication of this solution is to have the applicator spray the surface from one point, until the specified average thickness H is reached. Though this solution is unrealistic in terms of an actual implementation, the absolute minimum time necessary to achieve a specified average thickness is determined. This provides the lowest possible time bound for the time constrained minimum variation problem.

B. The Minimum Variation Problem

Given a parameterized spatial path $\mathbf{p}(\cdot)$, and an associated film accumulation rate function characterized by the mapping \mathbf{f} , the objective of the minimum variation problem is to determine the time profile that causes the variation in film thickness to be minimized, subject to achieving a specified average thickness H over the given surface.

B.1 Standard Formulation

With the standard approximation to the average thickness and the film thickness equations, the problem is formulated as

$$\min_{[\psi_1, \psi_2, \dots, \psi_N]} \{\tilde{\mathcal{V}}(\mathbf{p}(\cdot), \Psi, T)\} \quad (41)$$

$$\text{subject to } \mathbf{G}(\mathbf{p}(\cdot), \Psi, T) = \mathbf{H} \quad (42)$$

$$\text{and } 0 \leq \psi_i \leq L, \quad \forall i. \quad (43)$$

Substituting the standard approximations for variation in film thickness (Equation 10) and average film thickness (Equation 8) in the above formulation, the problem is expressed as

$$\min_{[\psi_1, \psi_2, \dots, \psi_N]} \left\{ \frac{1}{A_S} \int_S \left(\Delta \sum_{i=1}^N f(\mathbf{s}, \mathbf{p}(\psi_i)) \right)^2 d\mathbf{s} - \left(\frac{\Delta}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s} \right)^2 \right\} \quad (44)$$

$$\text{subject to } \frac{A}{A_S} \sum_{i=1}^N \int_S f(\mathbf{s}, \mathbf{p}(\psi_i)) d\mathbf{s} = \mathbf{H} \quad (45)$$

$$\text{and } 0 \leq \psi_i \leq L, \quad \forall i, \quad (46)$$

which is a nonlinear program in Ψ .

B.2 Alternate Formulation

Using the alternate approximations for the average thickness and variation in film thickness (Equations 23 and 24), the minimum variation problem can be expressed as a quadratic program in Γ . The problem is now posed as

$$\min_{[\tau_1, \tau_2, \dots, \tau_N]} \{\tilde{\mathcal{V}}_a(\mathbf{p}(\cdot), \Gamma, T)\} \quad (47)$$

$$\text{subject to } \tilde{\mathbf{G}}_a(\mathbf{p}(\cdot), \Gamma, T) = \mathbf{H} \quad (48)$$

$$\text{and } \tau_i \geq 0, \quad \forall i. \quad (49)$$

For convenience of notation, define a matrix \mathbf{P} , such that the $[i,j]$ -th element of \mathbf{P} is given by

$$P_{[i,j]} = \frac{1}{A_S} \int_S f(\mathbf{s}, \mathbf{p}_i) f(\mathbf{s}, \mathbf{p}_j) d\mathbf{s}. \quad (50)$$

Using this notation to express the objective function, the problem is expressed as

$$\min_{\Gamma} \{ \Gamma' P \Gamma - \Gamma' \mathbf{g} \mathbf{g}' \Gamma \}, \quad (51)$$

$$\text{subject to } \mathbf{g}' \Gamma = H \quad (52)$$

$$\text{and } \Gamma \geq 0. \quad (53)$$

The two constraints that are imposed on the solutions are the average thickness equality constraint and the constraint that the time values are positive. These can be written in the form of a single vector inequality given as

$$\underbrace{\begin{bmatrix} I_N \\ -\mathbf{g}' \\ \mathbf{g}' \end{bmatrix}}_K \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_N \end{bmatrix}}_{\Gamma} \geq \underbrace{\begin{bmatrix} 0 \\ -H \\ H \end{bmatrix}}_e \quad (54)$$

where I_N denotes the identity matrix. The constrained problem is written as a quadratic program in the following form:

$$\min_{\Gamma} \{ \Gamma(P - \mathbf{g} \mathbf{g}') \Gamma' \} \quad (55)$$

$$\text{subject to } K \Gamma \geq e. \quad (56)$$

This is a quadratic program in Γ that can be solved by standard quadratic programming routines. The condition for the solvability of this program to a global optimum is that the cost function should at least be positive semi-definite [10]. As the cost function in this case is the variation in film thickness (which is always non-negative), the matrix $\mathbf{P} - \mathbf{g} \mathbf{g}'$ is at least positive semi-definite. Therefore, the cost function is convex and a global optimum can be determined.

C. The Time Constrained Minimum Variation Problem

The time constrained minimum variation problem involves the addition of an upper bound constraint on painting time. With the standard approximations for the film thickness and average thickness functions, the painting time is determined by the product $\mathbf{N}\mathbf{A}$. The average thickness is also linearly proportional to \mathbf{A} (see Equation 45). Therefore \mathbf{A} must first be scaled to satisfy the equality constraint on the average film thickness. Then the inequality constraint on $\mathbf{N}\mathbf{A}$ can be verified.

With the alternate approximation, the constraint on painting time has to be introduced explicitly in the quadratic program described in the previous section. Nevertheless, as an upper bound on painting time is also a linear constraint, the quadratic structure of the program is not destroyed. The constraint can be appended as an extra row to the K and e matrices in Equation 56.

V. NUMERICAL STUDIES

In this section, numerical solutions to the optimization problems developed in the previous sections are derived by considering two different types of film accumulation rate functions. The first type of film accumulation rate function used, called an infinite range model, has the feature that its value actually goes to zero only as the distance between the applicator and the point on the surface tends to infinity. Examples of this type are the bivariate Cauchy distribution considered in [1], and the bivariate Gaussian distribution considered in [3]. The advantages of using these functions are: (1) the surface integrals can be readily evaluated (thus saving some computation time); and (2) the induced cost functions are quite smooth, which generally enhances the convergence properties of most nonlinear programming algorithms.

The second type of film accumulation rate function used, called a finite range

model, is a more accurate indicator of actual film accumulation rates, as the film accumulation rate function is zero for surface points that are outside a specific region surrounding the applicator's position. Such models can be specified based on empirical studies; an example is the model considered in [5]. In most finite range models, the integration of the film accumulation rate function must be done numerically, and the associated cost functions are not as smooth as those generated by the infinite range models. Thus, optimization studies involving these types of models tend to involve a higher computational burden. The particular finite range model used in the simulations is given in the next subsection, and the simulation results are summarized in the last two subsections.

Although two types of film accumulation rate models are used, the main purpose of the numerical studies is to illustrate the advantages of using the alternate formulation of over that of the standard formulation (described in Sections III and IV). The advantages of the proposed approach are shown with respect to both quality of results and CPU time.

A. A Finite Range Model for the Film Accumulation Rate Function

The model used for the rate of film accumulation in the finite range case is as described in [5]. The spray from the applicator is assumed to be shaped as a cone, and is symmetric about the axis of the nozzle. Within this spray cone, both the angle η from the central axis of the nozzle to the point on the surface and the height h of the nozzle from the surface impact the total rate of film accumulation at that point (for a given paint flow rate). In the simulations presented, the applicator is kept at a constant distance from the surface, thus, the value of the parameter h is a constant. The film accumulation rate model at a point on a flat plate at a height h from the tip

of the nozzle is proportional to

$$\frac{q(\eta)}{h^2}, \quad (57)$$

where $q(\eta)$ is given by,

$$q(\eta) = \begin{cases} 1 & \text{if } \eta < \alpha \\ 0.5 \left(1 + \cos \left(\frac{\pi(\eta-\alpha)}{\beta-\alpha} \right) \right) & \text{if } \alpha \leq \eta < \beta \\ 0 & \text{if } \eta \geq \beta \end{cases} . \quad (58)$$

The parameter α is the angle from the central axis of the nozzle to the inner spray boundary and β is the angle from the central axis of the nozzle to the outer spray boundary. Both α and β are assumed to be known constants, whose values depend on the particular characteristics of the applicator. Further details regarding the model are given in [5].

B. Comparative Optimization Studies

The optimization problems discussed in Section IV are solved based on the bivariate Cauchy distribution and the finite range film accumulation rate model presented above. A flat panel of dimensions $5\frac{1}{3} \times 5\frac{1}{3}$ is used as the surface on which a specified average thickness is to be achieved. The spray parameters α and β of the finite range model are chosen as 0.8 and 0.5, respectively, and the height of the applicator above the surface, h , is unity. The applicator is assumed to traverse a path that lies on a plane above, and parallel to, the panel. The analytical parameterization of the spatial path is given in the Appendix. The spatial path is shown in Figure 3. A value of $N = 74$ was used in all simulations.

The minimum time and minimum variation problems are solved for this example using the standard and alternate formulations. The standard formulation involves non-linear programming methods for determining a solution, and the IMSL routine BCONF [4] is used for this purpose. The minimum time and minimum variation problems using

the alternate formulation require the use of linear and quadratic programming routines. (Actually, as noted in the previous section, the minimum time solution is easily computed for the alternate formulation by finding the maximum element of the known vector \mathbf{g} .) The minimum variation problem is solved for the alternate formulation using the routine QPROG from the IMSL libraries.

The quality of the results and the CPU times required by the two formulations are outlined for comparison in Tables 1 and 2. The results of the simulation studies on the infinite range film accumulation rate model are given in Table 1. The corresponding results for the finite range model are given in Table 2. In both tables, NP (for nonlin-

Simulation studies	Minimum Time		Minimum Variation	
	NP	LP	NP	QP
Optimal Index	4.84	4.84	0.00164	0.00159
CPU Time	37.59	3.65	1262.2	12.35

Table 1: Comparison of solutions and CPU times for the infinite range film accumulation rate model.

ear programming) refers to the solutions obtained through the standard formulation. L/QP (for linear and quadratic programming) refers to the solutions of the alternate formulation. The average thickness is constrained to be one unit in all cases. The total painting time is the index in the minimum time problem, and the variation in film thickness is the index in the minimum variation problem. The CPU time is given in seconds. All simulations were done on a Sun SPARCstation 5. For both the infinite and finite range models (i.e., Tables 1 and 2), the minimum time solutions produced by NP and LP correspond to the applicator being positioned at one location over the entire time interval. As stated in the previous section, although this is an impractical solution, it does provide an absolute lower bound on painting time.

The characteristics of the solutions for the minimum variation problem produced by NP and QP for the finite range model are distinct. To illustrate this distinction, the

Simulation studies	Minimum Time		Minimum Variation	
	NP	LP	NP	QP
Optimal Index	15.94	15.94	0.2899	0.1822
CPU Time	2959.78	3.65	23679.9	63.73

Table 2: Comparison of solutions and CPU times for the finite range film accumulation rate model.

time profile solutions for the finite range case associated with NP and QP are shown in Figures 4 and 5, respectively. In addition to providing a superior performance index, the QP solution also has less abrupt changes than the NP solution (note the abrupt changes in Figure 4 that occur around 9, 19 and 31 units along the time axis). Because the spatial path $\mathbf{p}(\cdot)$ is parameterized by arc length, large accelerations in $\psi(t)$ correspond to large accelerations in the end-effector, which can be difficult to implement in practice. For more details on trajectory implementation, refer to [6, 8].

The computational effort required to determine the time profiles through NP and L/QP are also given in Tables 1 and 2. For the infinite range model, the time required by the L/QP formulation is between one and two orders of magnitude less than that of the NP formulation (Table I). The savings in computational time of L/QP increases to between two and three orders of magnitude in the finite range model (Table 2).

The initial conditions used for the minimum time problem were based on an appropriate discretization of a "stationary" time profile (i.e., $\psi(t) = 0$). The initial conditions used for the minimum variation problem were based on an appropriate discretization of a "constant speed" time profile (i.e., $\psi(t) = (L/T)t$). These initial conditions were used because they were found to produce superior performance indices for the NP (i.e., standard) approaches.

C. Process Optimization Studies with the Alternate Formulation and the Finite Range Model

To further illustrate the utility of the alternate formulation, two studies were conducted using the finite range model. The first study compares the performance of the minimum time and minimum variation solutions, in terms of variation in film thickness and total painting time. (For the minimum variation problem, no constraints were imposed on painting time.) The results of this study are summarized in Table 3. The performance of a constant speed trajectory is also tabulated for comparison¹. From

Type of solution	Variation	Painting time
Minimum time	12.23	15.94
Constant speed	0.1901	28.19
Minimum variation	0.1822	39.98

Table 3: A comparison of minimum time, constant speed, and minimum variation solutions (minimum time and minimum variation solutions were generated by the alternate formulation).

Table 3, it is seen that the minimum time solution has the highest variation in film thickness of all the solutions. The constant speed solution has a better variation in film thickness but requires more painting time. The trend continues for the minimum variation solution, where the variation is the least but the painting time is highest. Note that no constraints were placed on the painting time for this particular solution. The average thickness is constrained to be unity for all cases.

The second study, results of which are presented in Table 4, involves the comparison of minimum variation solutions for the finite range model, with constraints imposed on total time. Recall from Table 3 that the total painting time for the minimum variation

¹The constant speed solution corresponds to the case where equal units of time are spent along each segment of the spatial path. Thus, if the desired average thickness is H units, the time spent at each segment is given by

$$\tau_i = \frac{H}{1'g} \quad \forall i.$$

solution is more than that of the constant speed solution by about 40%. In industrial production lines, this difference may add up to a significant amount of "excess" finishing time. The motivation for imposing time constraints is to study the tradeoff between painting time and quality, as measured by the variation in film thickness. Two cases are presented in the study. First, the time taken for the constant speed case is used as an upper bound for painting time. Second, the time bound is lowered, so that the constant speed case is not a feasible solution.

Type of solution	Variance	Painting time
Painting time ≤ 28.19		
Constant speed	0.1901	28.19
Minimum variation	0.1843	28.19
Painting time ≤ 26		
Constant speed	—	—
Minimum variation	0.1852	26

Table 4: Comparison of solutions generated by the alternate formulation with constraints imposed on painting time. All cases use the finite range film accumulation rate model.

Some intuition about the effect of applying a time constraint to the minimum variation problem is gained by comparing the unconstrained minimum variation solution (Figure 5) to the minimum variation solution with a time constraint (Figure 6). For the unconstrained case, the applicator spends a significant amount of time at points along the curved portions of the trajectory where the rate of film accumulation on the surface is low. This is done in order to reduce the variation in film thickness near the edges of the surface (the curved portions of the spatial path are not directly above the plate). The time profile corresponding to the case where the painting time is constrained to 26 units is shown in Figure 6. The variation in film thickness is better than the constant speed time profile solution, and the painting time is less. For this case, a constant speed solution is impossible (i.e., the constant speed solution requires more than the allotted **26** units of time).

Preliminary work has been conducted for considering the case of curved surfaces. A general model for the rate of film accumulation for curved surfaces is derived in [7]. Simulation studies based on this model are currently underway.

VI. CONCLUSIONS

A class of optimal trajectory planning problems has been discussed with applications to automated spray coating. Conventional formulations for these applications generally yield nonlinear programming problems that are computationally expensive. The formulation developed in this paper is shown to yield linear or convex quadratic programming problems. The solution procedures are evaluated through simulation studies using two different models of film accumulation, and comparisons are made with earlier work from the literature. In the simulation studies, two separate optimization subroutines developed by IMSL Corporation (one specifically for quadratic programming problems, the other for general nonlinear programming problems) are used. It is shown that the quadratic programming problem associated with the proposed approach can be solved up to three orders of magnitude faster than the general nonlinear program required for the standard approach.

APPENDIX

An analytic parameterization of the spatial path shown in Figure 3 is given in this appendix. The parameterization is of the form $\mathbf{p} = [p_x(\lambda), p_y(\lambda)]'$, which represents the x and y coordinates of the applicator. The parameterization is in terms of the arc length, over the interval $[0, L]$. The length of the path is denoted by $L = 4\ell + \frac{3\pi d}{2}$, where, for the present study, $\ell = 5\frac{1}{3}$ and $d = 1\frac{7}{9}$. The expressions for $p_x(\lambda)$ and $p_y(\lambda)$ are defined by partitioning the interval $[0, 1]$ for λ into seven subintervals as follows:

- if $(0 \leq \lambda < \frac{1}{L}\ell)$

$$p_x(\lambda) = L\lambda$$

$$p_y(\lambda) = 3d$$

- if $(\frac{1}{L}\ell \leq \lambda < \frac{1}{L}(\ell + \frac{\pi d}{2}))$

$$p_x(\lambda) = \ell + \frac{d}{2} \cos[(\frac{1}{L}\ell - \lambda)\frac{2L}{d} + \frac{\pi}{2}]$$

$$p_y(\lambda) = \frac{5d}{2} + \frac{d}{2} \sin[(\frac{1}{L}\ell - \lambda)\frac{2L}{d} + \frac{\pi}{2}]$$

- if $(\frac{1}{L}(\ell + \frac{\pi d}{2}) \leq \lambda < \frac{1}{L}(2\ell + \frac{\pi d}{2}))$

$$p_x(\lambda) = 2\ell + \frac{\pi d}{2} - L\lambda$$

$$p_y(\lambda) = 2d$$

- if $(\frac{1}{L}(2\ell + \frac{\pi d}{2}) \leq \lambda < \frac{1}{L}(2\ell + \pi d))$

$$p_x(\lambda) = \frac{d}{2} \cos[\frac{2L}{d}(\lambda - \frac{1}{L}(2\ell + \frac{\pi d}{2})) + \frac{\pi}{2}]$$

$$p_y(\lambda) = \frac{3d}{2} + \frac{d}{2} \sin[\frac{2L}{d}(\lambda - \frac{1}{L}(2\ell + \frac{\pi d}{2})) + \frac{\pi}{2}]$$

- if $(\frac{1}{L}(2\ell + \pi d) \leq \lambda < \frac{1}{L}(3\ell + \pi d))$

$$p_x(\lambda) = L\lambda - 2\ell - \pi d$$

$$p_y(\lambda) = d$$

- if $(\frac{1}{L}(3\ell + \pi d) \leq \lambda < \frac{1}{L}(3\ell + \frac{3\pi d}{2}))$

$$p_x(\lambda) = \ell + \frac{d}{2} \cos[(\frac{1}{L}(3\ell + \pi d) - \lambda)\frac{2L}{2} + \frac{\pi}{2}]$$

$$p_y(\lambda) = \frac{d}{2} + \frac{d}{2} \sin[(\frac{1}{L}(3\ell + \pi d) - \lambda)\frac{2L}{2} + \frac{\pi}{2}]$$

- if $(\frac{1}{L}(3\ell + \frac{3\pi d}{2}) \leq \lambda < \frac{1}{L}(4\ell + \frac{3\pi d}{2}) = 1)$

$$p_x(\lambda) = 4\ell + \frac{3\pi d}{2} - L\lambda$$

$$p_y(\lambda) = 0.$$

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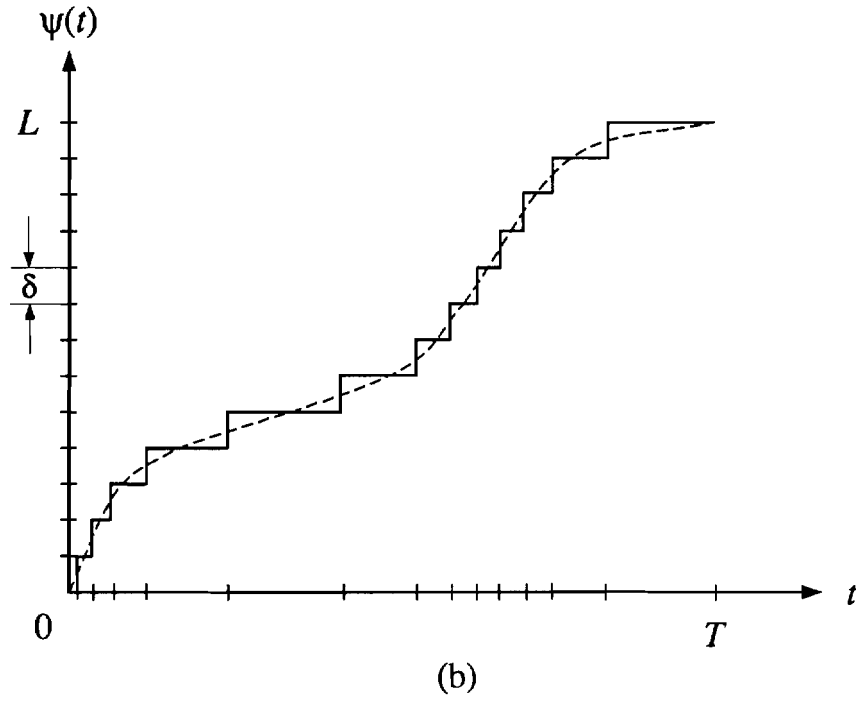
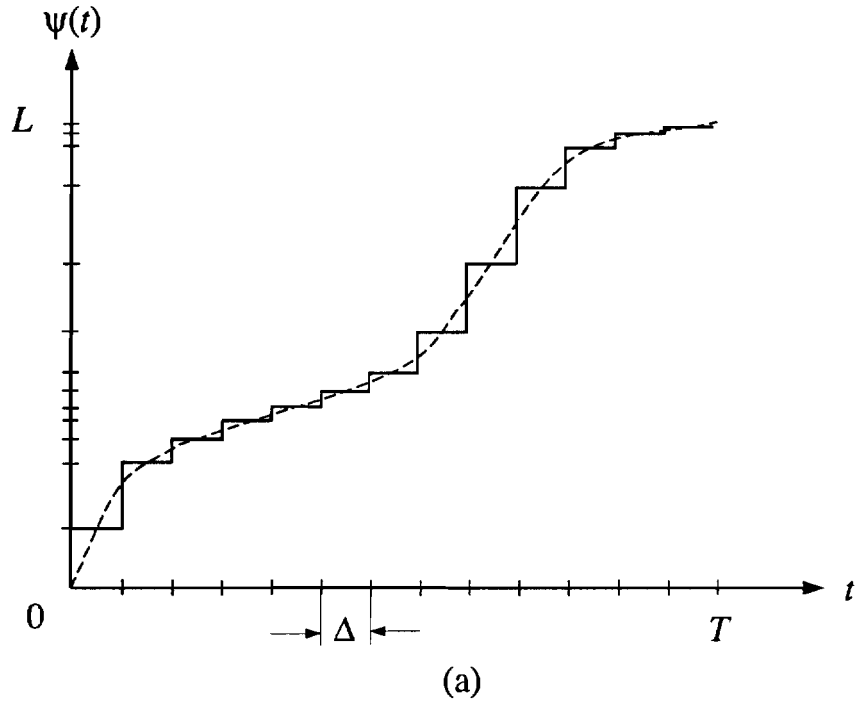


Figure 1: Two piece-wise constant approximations of a monotone function: (a) represents a constant width discretization along the time axis and (b) represents a constant width discretization along the spatial axis.

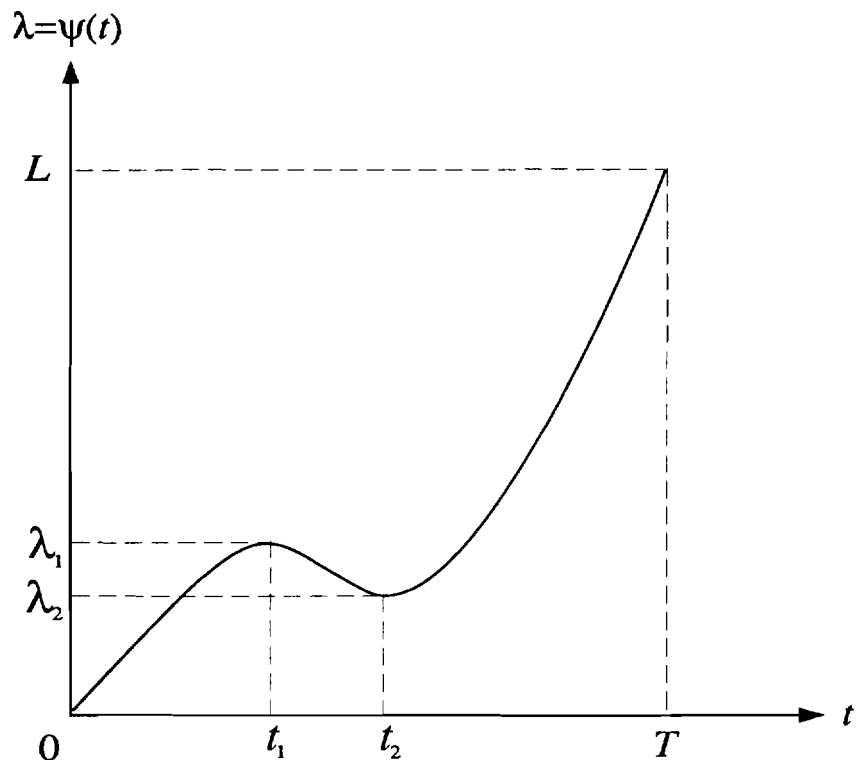


Figure 2: Non-monotone function used in the proof of Theorem 1.

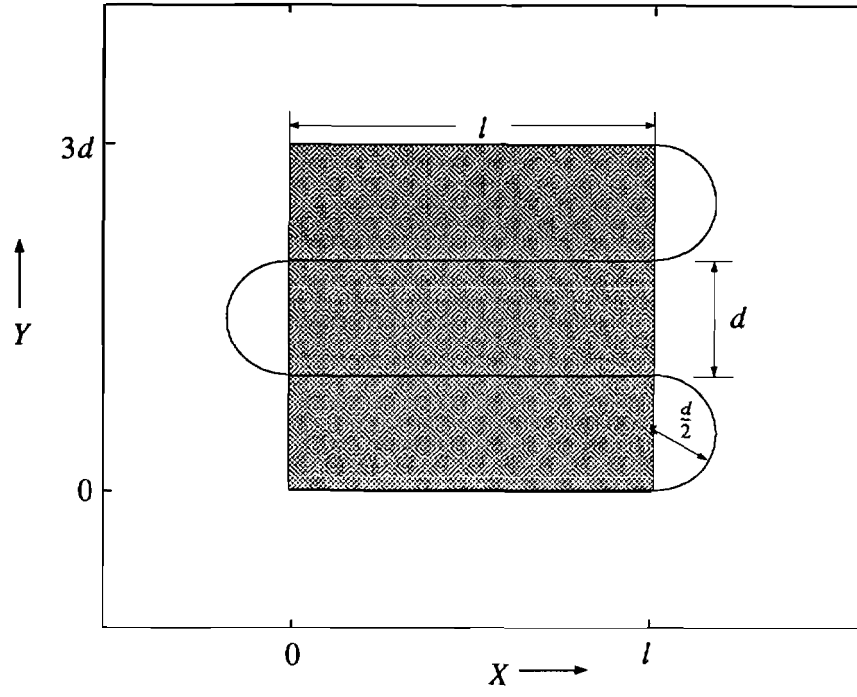


Figure 3: The spatial path chosen for the simulation studies conducted in this paper. The path is used to traverse over a square plate, indicated by the shaded area.

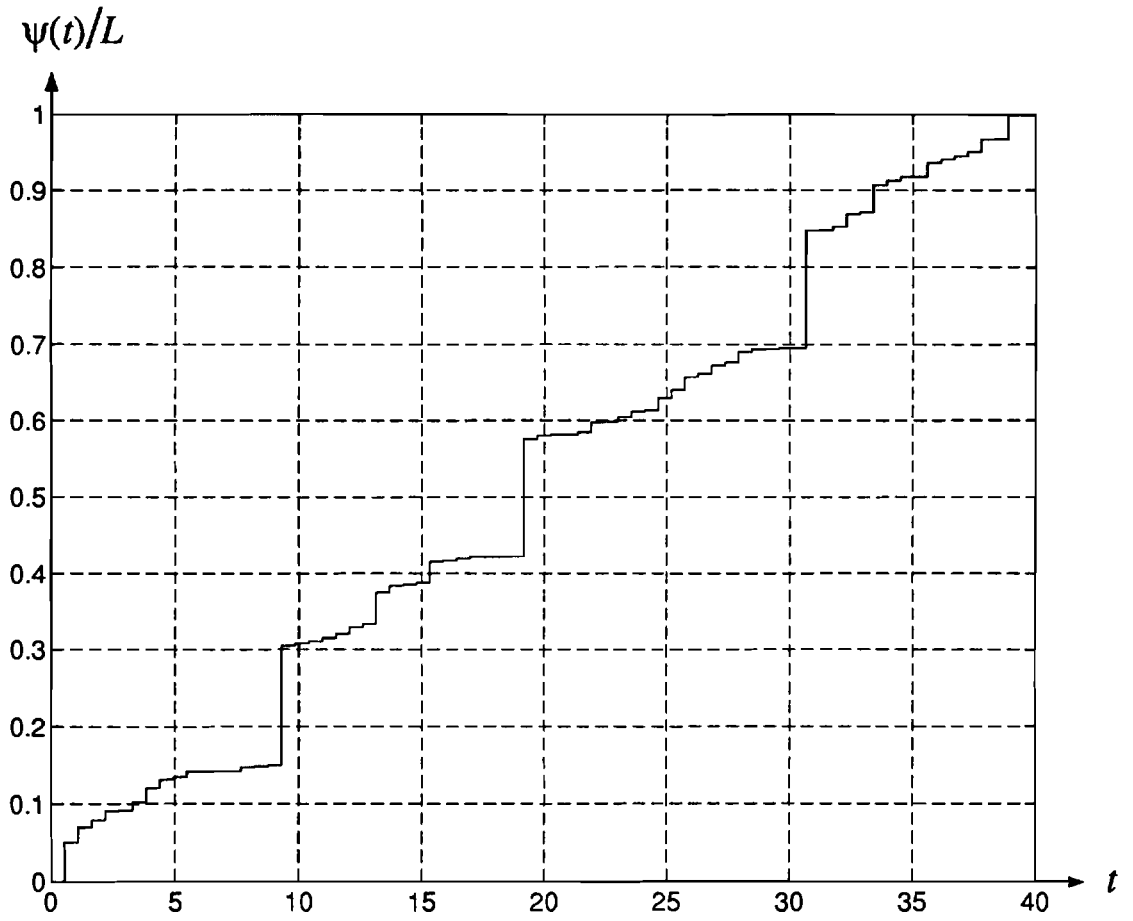


Figure 4: The unconstrained minimum variation solution for the finite range film accumulation rate model using the standard formulation. The solution was obtained by solving a nonlinear programming problem.

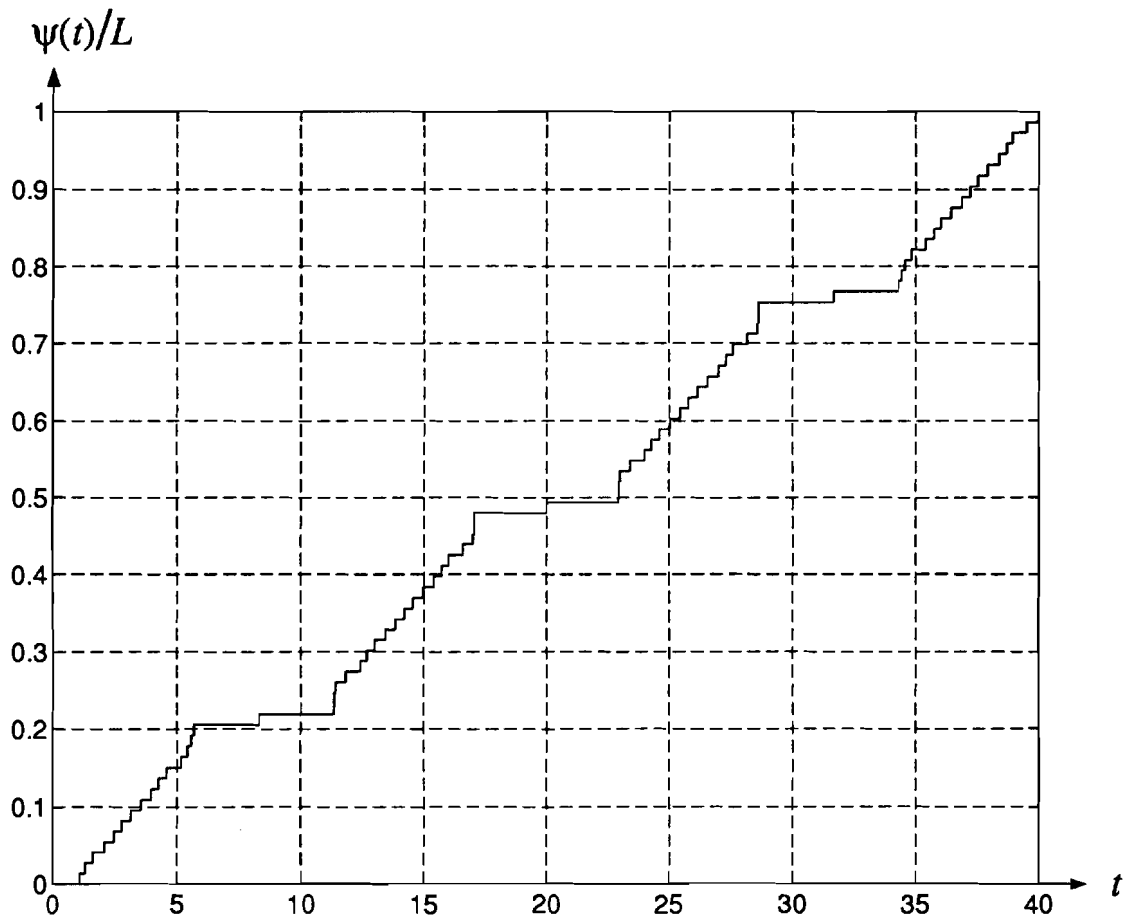


Figure 5: The unconstrained minimum variation solution for the finite range film accumulation rate model using the alternate formulation. The solution was obtained by solving a quadratic programming problem.

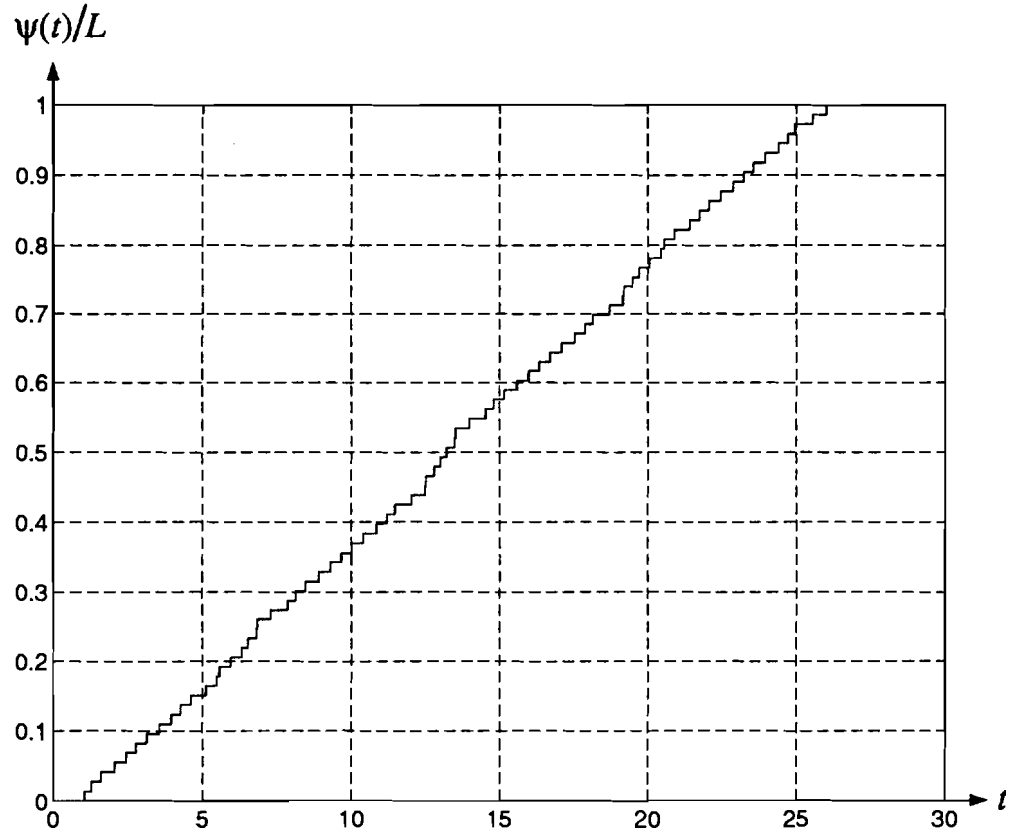


Figure 6 A minimum variation solution for the finite range film accumulation rate model with painting time constrained to be less than or equal to 26 units. The amount of time spent at the three curved parts of the spatial path is reduced (compared to the unconstrained case of Figure 5), but the amount of time spent at the other points are roughly the same as the unconstrained solution of Figure 5.